12. SYLOW SUBGROUPS S_n

§12.1. Wreath Products

Suppose G is any group and $H \leq \mathbf{S}_n$. The **wreath product** is:
$G \int H = \{[(g_1, g_2, \ldots, g_n); \pi] \mid \text{each } g_i \in G \text{ and } \pi \in H\}$. Often we leave out the inner parentheses and write $[g_1, g_2, \ldots, g_n; \pi]$ instead of $[(g_1, g_2, \ldots, g_n); \pi]$.

If G is finite, the order of $G \int H$ is $|G|^n.|H|$.

We make this into a group by defining:

$$[x_1, x_2, \ldots, x_n; \alpha]. [y_1, y_2, \ldots, y_n; \beta]$$
$$= [x_1 y_{\alpha(1)}, x_2 y_{\alpha(2)}, \ldots, x_n y_{\alpha(n)}; \alpha\beta].$$

We can write this more compactly as:

$$[(x_i);\alpha].[(y_i);\beta) = [(x_i y_{\alpha(i)});\alpha\beta).$$

We can check that this is associative as follows:

$$([(x_i);\alpha].[(y_i);\beta)]).[(z_i);\gamma] = [(x_i y_{\beta(i)});\alpha\beta].[(z_i);\gamma]$$
$$= [((x_i y_{\beta(i)}) z_{\alpha\beta(i)});(\alpha\beta)\gamma]$$
$$= [(x_i y_{\beta(i)} z_{\alpha\beta(i)});\alpha\beta\gamma]$$
$$= [(x_i);\alpha].([(y_i);\beta)].[(z_i);\gamma])$$

The identity of $G \int H$ is $[1, 1, \ldots, 1;1]$ where the last 1 is the identity of H and the preceding ones are the identity of G.

Example 1: $|\mathbf{C}_2 \int \mathbf{S}_3| = 2^3.6 = 48$. Let $C_2 = \langle \alpha \mid \alpha^2 \rangle$.
The elements of $\mathbf{C}_2 \int \mathbf{S}_3$ are:

[1, 1, 1; I]	[1, 1, 1; (123)]	[1, 1, 1; (132)]
[1, 1, α; I]	[1, 1, α; (123)]	[1, 1, α; (132)]
[1, α, 1; I]	[1, α, 1; (123)]	[1, α, 1; (132)]
[1, α, α; I]	[1, α, α; (123)]	[1, α, α; (132)]
[α, 1, 1; I]	[α, 1, 1; (123)]	[α, 1, 1; (132)]
[α, 1, α; I]	[α, 1, α; (123)]	[α, 1, α; (132)]
[α,α, 1; I]	[α,α, 1; (123)]	[α,α, 1; (132)]
[α,α, α; I]	[α,α, α; (123)]	[α,α, α; (132)]
[1, 1, 1; (12)]	[1, 1, 1; (23)]	[1, 1, 1; (13)]
[1, 1, α; (12)]	[1, 1, α; (23)]	[1, 1, α; (13)]
[1, α, 1; (12)]	[1, α, 1; (23)]	[1, α, 1; (13)]
[1, α, α; (12)]	[1, α, α; (23)]	[1, α, α; (13)]
[α, 1, 1; (12)]	[α, 1, 1; (23)]	[α, 1, 1; (13)]
[α, 1, α; (12)]	[α, 1, α; (23)]	[α, 1, α; (13)]
[α,α, 1; (12)]	[α,α, 1; (23)]	[α,α, 1; (13)]
[α,α, α; (12)]	[α,α, α; (23)]	[α,α, α; (13)]

An example of multiplication is:

$$[\alpha, 1, \alpha; (123)].[\alpha, \alpha, 1; (23)]$$
$$= [(\alpha, 1, \alpha).(\alpha, 1, \alpha); (123).(23)]$$
$$= [\alpha^2, 1, \alpha^2; (13)]$$
$$= [1, 1, 1; (13)].$$

We can extend the concept of the wreath product G ∫ H to cases where H, instead of being a permutation group, acts on a set {1, 2, …, *n*} but where more than one element induces the same permutation. In the case where H acts on {1} trivially, meaning that $h(1) = 1$ for all $h \in$ H, G ∫ H becomes just the direct product G × H.

Example 2: Find the orders of the wreath products (a) $\mathbf{S}_3 \int \mathbf{S}_4$ and (b) $\mathbf{S}_4 \int \mathbf{S}_3$.

Solution: (a) $\mathbf{S}_3 \int \mathbf{S}_4$ consists of a direct product of 4 copies of $\mathbf{S}_3$ extended by $\mathbf{S}_4$.

Its order is therefore $(3!)^4.4! = 6^4.24 = 31{,}104$.

(b) $\mathbf{S}_4 \int \mathbf{S}_3$ consists of a direct product of 3 copies of $\mathbf{S}_4$ extended by $\mathbf{S}_3$.

Its order is therefore $(4!)^3.3! = 24^3.6 = 82{,}944$.

Example 3:

Find the largest order of any element of $\mathbf{S}_3 \int \mathbf{S}_3$.

Solution:

The largest order of any element of $\mathbf{S}_3 \times \mathbf{S}_3 \times \mathbf{S}_3$ is the same as the largest order of any element of $\mathbf{S}_3$, namely 3.

If $x = [a, b, c]$ then x has order 3 at most.

If $x = [a, b, c](\times\times)$, where $(\times\times)$ is any 2-cycle, then $x^2 \in S_3 \times S_3 \times S_3$ and so x has order 6 at most.

If $x = [a, b, c](123)$ then

$$x^2 = [a, b, c](123).[a, b, c](123)$$
$$= [a, b, c][b, c, a](132)$$
$$= [ab, bc, ac](132).$$

$$\therefore\ x^3 = [a, b, c](123)\ [ab, bc, ac](132)$$
$$= [a, b, c][bc, ac, ab]$$
$$= [abc, abc, abc]$$

So x has order at most 9 and if $a = b = I$ and $c = (123)$ then x in fact has order 9.
Hence the maximum order of any element of $\mathbf{S}_3 \int \mathbf{S}_3$ is 9.

§12.2. Wreath Products Where the Action is not Specified

If H is not specified as a permutation group it is assumed, for the purposes of the wreath product, to act according to the regular representation. In other words we code the elements of H as 1, 2, 3, .., n where $n = |H|$.

If $h \in H$ has code $\Gamma(h)$ then for $h \in H$, $h(k) = \Gamma(\Gamma^{-1}(k)h)$. For example, the action of $b \in H$ on the symbol 3 can be found by taking the element $a \in$ of H with code 3, multiplying it on the right by b and then looking up the code for ab.

Example 4: $\mathbf{C}_2 \int \mathbf{D}_6 = \{[g_1, g_2, g_3, g_4, g_5, g_6; h]$ where each $g_i \in \mathbf{C}_2$ and $h \in \mathbf{D}_6$. This is a larger group than $\mathbf{C}_2 \int \mathbf{S}_3$, notwithstanding the fact that $\mathbf{D}_6 \cong \mathbf{S}_3$. The action of the group H is part of the definition.

If we choose to use the regular action the actual wreath products will depend on the numbers we assign to the elements of H, though they will all be isomorphic.

Example 5: Let $G = D_8 = \langle A, B | A^4, B^2, BA = A^{-1}B\rangle$ and $H = C_3 = \langle C \mid C^3\rangle$. Then $G \int H$ has order $8^3.3 = 1536$. Let's code H by coding 1, C, C^2 as 1, 2, 3 respectively.
Then C(1) = 2, C(2) = 3 and C(3) = 1.

An example of multiplication is:

$[AB, A, I; C^2].[A^3B, A^2, B; C^2]$

$= [(AB, A, I). (B, A^3B, A^2);C^4)]$

$= [AB^2, A^4B, A^2; C]$

$= [A, B, A^2; C].$

I've tried to describe wreath products in complete generality, but for what follows we just need wreath products built up from cyclic groups. However we'll have wreath products of wreath products.

The associative law fails for wreath products. By this I mean that (G ∫ H) ∫ K is not usually isomorphic to G ∫ (H∫ K).

But we associate from the left and write G ∫ H ∫ K to mean the iterated wreath product (G ∫ H) ∫ K.

We write G ∫ G ∫ … ∫ G (with *n* terms) as $G^{[n]}$. We will also have iterated direct products and we'll write G^n to denote G × G × … × G (with *n* factors).

Example 6: The order of $C_2^{(4)} \times C_3^5$ is

Here is how we can build it up: $2^{15}.3^5$.

GROUP	ORDER
$\mathbf{C}_2$	2
$\mathbf{C}_2^{[2]} = \mathbf{C}_2 \int \mathbf{C}_2$	$2^2.2 = 2^3$
$\mathbf{C}_2^{[3]} = \mathbf{C}_2 \int \mathbf{C}_2 \int \mathbf{C}_2$	$(2^3)^2.2 = 2^7$
$\mathbf{C}_2^{[4]} = \mathbf{C}_2 \int \mathbf{C}_2 \int \mathbf{C}_2 \int \mathbf{C}_2$	$(2^7)^2.2 = 2^{15}$
$\mathbf{C}_3$	3
$\mathbf{C}_3^5$	3^5

§12.3. Sylow Subgroups of S_n

Example 7: Find Sylow subgroups of $\mathbf{S}_{11}$ for each prime.
Solution: The set of prime divisors of 11! is

$$\Omega = \{2, 3, 5, 7, 11\}.$$

***p* = 11:** $\langle(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11)\rangle$ has order 11 and so is a Sylow 11-subgroup of $\mathbf{S}_{11}$.

***p* = 7:** $\langle(1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)\rangle$ has order 7 and so is a Sylow 7-subgroup of $\mathbf{S}_{11}$.

***p* = 5:** If $A = \langle(1\ 2\ 3\ 4\ 5)\rangle$ and $B = \langle(6\ 7\ 8\ 9\ 10)\rangle$ then $A \times B$ has order 5^2, which is the largest power of 5 dividing 11! It is therefore a Sylow 5-subgroup of $\mathbf{S}_{11}$.

***p* = 3:** Clearly $\mathbf{S}_{11}$ has a subgroup of order 3^3 that is isomorphic to $\mathbf{C}_3{}^3$. But the largest power of 3 that divides 11! is 3^4. So this is not a Sylow 3-subgroup.

Let $H = \langle(1\ 2\ 3)\rangle \times \langle(4\ 5\ 6)\rangle \times \langle(7\ 8\ 9)\rangle$. We can extend H by an element of order 3 that takes each block of three to the next, with the last going to the first. More specifically, let $x = (1\ 4\ 7)(2\ 5\ 8)(3\ 6\ 9)$.

Then $x^{-1}(1\ 2\ 3)x = (2\ 5\ 8)$,

$x^{-1}(2\ 5\ 8)x = (3\ 6\ 9)$ and

$x^{-1}(3\ 6\ 9)x = (1\ 2\ 3)$

The group generated by H and x consists of all elements of $\mathbf{S}_{11}$ of the form hx^r where $h \in H$ and $r \in \{0, 1, 2\}$. Clearly it has order 3^4 and so is a Sylow 3-subgroup.

***p* = 2:** The largest power of 2 that divides 11! is 2^8.

We can easily find a subgroup of $\mathbf{S}_{11}$ isomorphic to $\mathbf{C}_2{}^5$ such as H = ⟨(1 2)⟩ × ⟨(3 4)⟩ × ⟨(5 6)⟩ × ⟨(7 8)⟩ × ⟨(9 10)⟩. This has order 2^5 but the largest power of 2 dividing 11! is 2^8.

Partition {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11} into pairs, as far as we can. We get:
{1, 2}, {3, 4}, {5, 6}. {9, 8}, {9, 10}, with {11} left over. We can take a $\mathbf{C}_2$ for each of these to get $\mathbf{C}_2{}^5$ which has order 2^5.

Now partition these pairs into pairs of pairs, as far as we can.
We get {{1, 2}, {3, 4}}, {{5, 6}, {7, 8}}, with {9, 10} left over.

Now on each of these pairs of pairs we can construct a group isomorphic to $\mathbf{C}_2{}^{[2]}$ of order 2^3. If we now take $\mathbf{C}_2{}^{[3]}$ we get a group of order $(2^3)^2.2 = 2^7$.

But we had {9, 10} left over, on which we can form another $\mathbf{C}_2$. The Sylow 2-subgroup is thus $\mathbf{C}_2{}^{[3]} \times \mathbf{C}_2$, which has order 2^8.

We'll assume that wreath products take precedence over direct products, so that A × B ∫ C will be assumed to mean A × (B ∫ C). We can therefore write the Sylow 2-subgroups of $\mathbf{S}_{11}$, up to isomorphism, as

$$\mathbf{C}_2{}^{[3]} \times \mathbf{C}_2.$$

The Sylow subgroups of $\mathbf{S}_{11}$ are therefore:

p	Sylow p-subgroup
11	$\mathbf{C}_{11}$
7	$\mathbf{C}_7$
5	$\mathbf{C}_5{}^{[2]}$
3	$\mathbf{C}_3{}^{[2]}$
2	$\mathbf{C}_2{}^{[4]}$

Example 8: Find the Sylow 5-subgroups of $\mathbf{S}_{57}$.
Solution: Writing 57 in base 5 we get 212. This is because we can write 57 as $5^2.2 + 5.1 + 1.2$.
We can therefore partition {1, 2, ..., 58} as

$$\{1, 2, ..., 25\}, \{26, 27, ..., 50\}, \{51, 52, 53, 54, 55\}, \{56\}, \{57\}.$$

On each block of 25 symbols we can construct $\mathbf{C}_5{}^{[2]}$ and on the block of 5 we can construct $\mathbf{C}_5$. So the Sylow 5-subgroups of $\mathbf{S}_{57}$ are therefore isomorphic to

$$(\mathbf{C}_5{}^{[2]})^2 \times \mathbf{C}_5.$$

Theorem 1 (COOPER): If p is prime and

$$n = a_r p^r + a_{r-1} p^{r-1} + ... + a_1 p + a_0,$$

where for each i, $0 \le a_i < p$, the Sylow p-subgroups of $\mathbf{S}_n$ are isomorphic to

$$\mathbf{C}_p(r)a_r \times \mathbf{C}_p(r-1)a_{r-1} \times ... \times \mathbf{C}_p(2)a_2 \times \mathbf{C}_p a_1.$$

Proof: The existence of a subgroup of $\mathbf{S}_n$ the above form should be now obvious. I omit the proof that its order is the highest power of p that divides n!

Example 9: Find Sylow *p*-subgroups of $\mathbf{S}_{30}$ for $p \leq 7$ and their orders.

Solution:

p	Sylow p-subgroup	order
7	$\mathbf{C}_7{}^4$	7^4
5	$\mathbf{C}_5{}^{[2]} \times \mathbf{C}_5$	5^7
3	$\mathbf{C}_3{}^{[3]} \times \mathbf{C}_3$	3^{14}
2	$\mathbf{C}_2{}^{[4]} \times \mathbf{C}_2{}^{[3]} \times \mathbf{C}_2{}^{[2]} \times \mathbf{C}_2$	2^{26}

EXERCISES FOR CHAPTER 12

Exercise 1: For each of the following statements determine whether it is true or false.
(1) $\mathbf{C}_2 \wr \mathbf{C}_2 \cong \mathbf{D}_8$.
(2) $G \wr H \cong H \wr G$ for all groups G, H.
(3) The Sylow 3-subgroup of $\mathbf{S}_{14}$ is isomorphic to

$$\mathbf{S}_3^{[2]} \times \mathbf{S}_3.$$

(4) The Sylow 2-subgroup of $\mathbf{S}_{14}$ has order 2^{10}.
(5) The wreath product of two soluble groups is soluble.
(6) The Sylow 2-subgroup of $\mathbf{S}_{20}$ is isomorphic to

$$\mathbf{C}_2^{[4]} \times \mathbf{C}_2^{[2]} \times \mathbf{C}_2$$

Exercise 2: Show that $\mathbf{C}_3 \wr \mathbf{S}_2 \cong \mathbf{D}_6 \times \mathbf{C}_3$.

Exercise 3: Find all the Sylow subgroups of $\mathbf{S}_{27}$.

SOLUTIONS FOR CHAPTER 12

Exercise 1:
(1) FALSE $\mathbf{C}_2 \wr \mathbf{C}_2 = \mathbf{C}_2 \wr \mathbf{S}_2$

$$= \{(x, y; \pi) \mid x, y \in \langle \alpha \mid \alpha^2 \rangle, \pi \in \mathbf{S}_2\}$$

The elements of $\mathbf{C}_2 \wr \mathbf{S}_2$ are:

$(1, 1; I)$	$(1, \alpha; I)$	$(\alpha, 1; I)$	$(\alpha, \alpha; I)$
$(1, 1; (12))$	$(1, \alpha; (12))$	$(\alpha, 1; (12))$	$(\alpha, \alpha; (12))$

Clearly $(1, 1; I)$ is the identity.

Any element of the form $(x, y; \mathrm{I})$ has order 2.
Any element of the form $(x, x; (12))$ has order 2.
The remaining two elements have order 4.
It is isomorphic to $\mathbf{D}_8$ with:

$$(\alpha, 1; (12)) \to \mathrm{A} \text{ and } (1, \alpha; \mathrm{I}) \to \mathrm{B}.$$

(2) FALSE: If $|\mathrm{G}| = m$ and $|\mathrm{H}| = n$ then $|\mathrm{G} \int \mathrm{H}| = m^n.n$.
(3) FALSE: $\mathbf{S}_3^{[2]} \times \mathbf{S}_3$ will contain elements of order 3.
(4) TRUE
(5) TRUE
(6) FALSE: The largest power of 2 dividing 20! is 2^{18} while the order of $\mathbf{C}_2^{[4]} \times \mathbf{C}_2^{[2]} \times \mathbf{C}_2$ is 2^{19}.

Exercise 2:

Let $\mathbf{C}_3 = \langle \alpha \mid \alpha^3 \rangle$. The elements of $\mathbf{C}_3 \int \mathbf{S}_2$, and their orders, are:

$(1, 1; \mathrm{I})$	**1**	$(1, \alpha; \mathrm{I})$	**3**	$(1, \alpha^2; \mathrm{I})$	**3**
$(\alpha, 1; \mathrm{I})$	**3**	$(\alpha, \alpha; \mathrm{I})$	**3**	$(\alpha, \alpha^2; \mathrm{I})$	**3**
$(\alpha^2, 1; \mathrm{I})$	**3**	$(\alpha^2, \alpha; \mathrm{I})$	**3**	$(\alpha^2, \alpha; \mathrm{I})$	**3**
$(1, 1; (12))$	**2**	$(1, \alpha; (12))$	**6**	$(1, \alpha^2; (12))$	**6**
$(\alpha, 1; (12))$	**6**	$(\alpha, \alpha; (12))$	**6**	$(\alpha, \alpha^2; (12))$	**2**
$(\alpha^2, 1; (12))$	**6**	$(\alpha^2, \alpha; (12))$	**2**	$(\alpha^2, \alpha; (12))$	**6**

For example:

$$\begin{aligned}(\alpha, 1; (12)).\,(\alpha, 1; (12)) &= (\alpha, 1)(12)(\alpha, 1)(12)\\ &= (\alpha, 1)(1, \alpha)(12)^2\\ &= (\alpha, \alpha; \mathrm{I}) \text{ which has order 3.}\end{aligned}$$

and so $(\alpha, 1; (12))$ has order 6.
Also $(\alpha, \alpha^2; (12)).\,(\alpha, \alpha^2; (12)) = (\alpha, \alpha^2)(12)(\alpha, \alpha^2)(12)$

$= (\alpha, \alpha^2)(\alpha^2, \alpha)(12)^2$
$= (\alpha^3, \alpha^3; I) = (1, 1; I)$

and so $(\alpha, \alpha^2; (12))$ has order 2.

Now the elements of order 2 in $\mathbf{D}_6 \times \mathbf{C}_3$ generate the $\mathbf{D}_6$.
Let B = (1, 1; (12)).

$(1, 1; (12)). (\alpha, \alpha^2; (12)) = (1, 1)(12)(\alpha, \alpha^2)(12)$
$= (1, 1)(\alpha^2, \alpha)(12)^2$
$= (\alpha^2, \alpha; I)$ which has order 3.

This element of order 3 must be one of the two inside the $\mathbf{D}_6$, so let $A = (\alpha^2, \alpha; I)$. We can now check that the group generated by A, B is:

$$\langle A, B \mid A^3, B^2, BA = A^{-1}B\rangle \cong \mathbf{D}_6.$$

We now need to choose an element of order 3 to be the generator of the $\mathbf{C}_3$. It has to be in the centre and so, with a little bit of experimenting we choose $C = (\alpha, \alpha; I)$.

Exercise 3:

p	**Sylow *p*-subgroup**	**order**
2	$\mathbf{C}_2^{[4]} \times \mathbf{C}_2^{[3]} \times \mathbf{C}_2$	2^{23}
3	$\mathbf{C}_3^{[3]}$	3^{13}
5	$\mathbf{C}_5^{[2]}$	5^6
7	$\mathbf{C}_7^{[3]}$	7^3
11	$\mathbf{C}_{11}^{[2]}$	11^2
13	$\mathbf{C}_{13}^{[2]}$	13^2

For p = 17, 19 and 23 the Sylow p-subgroup $\cong \mathbf{C}_p$.